ON THE STATIONARY DISTRIBUTION OF MUTUALLY ATTRACTING PARTICLES

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The well known barometric formula of Laplace

$$p = p_0 \exp\left[-\frac{mgh}{kT}\right] \tag{1}$$

is a particular, and by no means complete, example of the application of Boltzmann's general formula

$$\rho = \rho_0 \exp\left[-\frac{\Pi}{kT}\right] \tag{2}$$

where Π is the potential energy of the particle and T the absolute temperature, identical for all the parts of the region, to which the distribution of the density ρ is related. However, the state of the system, though stationary in the thermal sense, need not necessarily be isothermal.

Below we will investigate an infinite body whose particles are subjected only to forces of mutual attraction in accordance with Newton's law. The equation of state need not be specified, it can be arbitrary. In the end some state of the system will be established, and it will necessarily possess spherical symmetry (we assume an infinite number of particles). A nonlinear integral equation is derived, connecting distribution of temperature with distribution of density. Knowing the distribution of the one, the distribution of the other can be obtained from this equation. For the Clapeyron state the problem can easily be carried through to completion. The above formulas (1) and (2) are obtained as particular cases for the isothermal state.

Let us investigate a gaseous body consisting of a great number of particles each of mass m mutually attracted according to the law

$$f = \gamma \frac{m^2}{l^2} \qquad \left([\gamma] = \frac{\text{dyn cm}^2}{\text{g}^2}\right) \tag{3}$$

(here γ is the gravitational constant). In formula (3) one would like to write the more general l^n instead of l^2 for Newtonian attraction with a view to applying this investigation to the micro-phenomena of molecular and atomic order also, for instance to the case of Van-der-Waals action and exchange forces. In that case, however, we cannot use that particular feature of Newtonian attraction of a spherical layer which is the basis of the derivation that follows. Such a generalization would complicate the calculations.

On achieving a stationary state, the gas inevitably assumes a form of spherical symmetry around the center of mass. This state will be considered.

Let O be the center of mass. Let us envisage a solid angle $d\omega$ with its apex at O and a volume element in it between the spherical surfaces of radii α and $\alpha + d\alpha$.

If the gas density in this volume is $\rho(\alpha)$, then its mass is

$$\rho(\alpha)\,\alpha^2\,d\omega\,d\alpha \tag{4}$$

The resultant action of all forces of attraction on that portion of the gas is as if it were attracted only by the gas contained in a sphere of radius α, and as if the (attracting) gas were concentrated in the center O of the sphere.

The attracting mass is equal to

$$\int_0^{\alpha} \rho(\beta)\,4\pi\beta^2\,d\beta \tag{5}$$

On the basis of (3), the force of attraction between masses (4) and (5) is

$$\gamma\rho(\alpha)\,d\alpha\,d\omega \int_0^{\alpha} \rho(\beta)\,4\pi\beta^2\,d\beta \tag{6}$$

Now imagine a spherical surface with center in O and radius r, and consider the portion contained within the solid angle $d\omega$. The area of this portion is $r^2 d\omega$. Above $r^2 d\omega$ is the totality of elementary volumes of infinite extent, each with a mass given by (4) for $\alpha \geqslant r$, and each being attracted to O by a force (6). These forces add up to create the force of pressure on area $r^2 d\omega$, equal to

$$\int_r^{\infty} \gamma\rho(\alpha)\,d\alpha\,d\omega \int_0^{\alpha} \rho(\beta)\,4\pi\beta^2 d\beta \tag{7}$$

Dividing (7) by the area $r^2 d\omega$, we obtain the gas pressure $p(r)$ at a distance r from the center of mass:

$$p(r)=\frac{\gamma}{r^2}\int_r^{\infty}\rho(\alpha)\,d\alpha\int_0^{\alpha}\rho(\beta)\,4\pi\beta^2\,d\beta \tag{8}$$

On the other hand, assume that the state of the gas is described by the equation

$$p=\varphi(\rho,T) \tag{9}$$

where, on account of the stationary state and symmetry ρ and T can only depend on r.

Substituting (9) in (8) we obtain

$$r^2\varphi[\rho(r),T(r)]=\gamma\int_r^{\infty}\rho(\alpha)\,d\alpha\int_0^{\alpha}\rho(\beta)\,4\pi\beta^2\,d\beta \tag{10}$$

For a given temperature distribution the distribution of density $\rho(r)$ can be obtained from this nonlinear integral equation, and vice versa.

In the case of a state according to the Clapeyron equation, for instance (10) becomes

$$T(r)\,\rho(r)\,4\pi r^2=\frac{\gamma m}{k}\int_r^{\infty}\frac{\rho(\alpha)\,4\pi\alpha^2\,d\alpha}{\alpha^2}\int_0^{\alpha}\rho(\beta)\,4\pi\beta^2\,d\beta \tag{11}$$

Let us note that

$$\int_0^{z}\rho(\zeta)\,4\pi\zeta^2\,d\zeta=M(z) \tag{12}$$

is the mass of gas in a sphere of radius z. It follows from (12) that

$$\rho(z)\,4\pi z^2=M'(z) \tag{13}$$

Bearing in mind (12) and (13), we can present (11) as

$$T(r)\,M'(r)=\frac{\gamma m}{k}\int\frac{M'(\alpha)\,M(\alpha)}{\alpha^2}\,d\alpha \tag{14}$$

Differentiating, we obtain

$$[T(r)\,M'(r)]'=-\frac{\gamma m}{k}\frac{M(r)\,M'(r)}{r^2}\quad\text{or}\quad TM''+T'M'=-\frac{\gamma m}{k}\frac{MM'}{r^2} \tag{15}$$

This ordinary differential equation, of second-order with respect to M and of first order with respect to T, connects the mass distribution with the temperature distribution of an ideal gas in a stationary thermal state. Either of these two distributions can be determined from the other.

With respect to M, equation (15) is reduced to a type of Liouville equation which cannot be integrated in finite form. But let us imagine that the gas sphere has a kernel of radius r_0 and mass M_0 so large that the outer gas mass will appear infinitely small in comparison. Equation

(15) can then be presented in the form

$$TM'' = -\left[\frac{a}{r^2} + T'\right] M', \qquad \frac{\gamma M_0 m}{k} = a, \qquad [a] = \text{deg cm} \tag{16}$$

For a given $T(r)$, this equation is easily solved. Actually it is sufficient to obtain from it $M'(r)$, and then use (13) to find $\rho(r)$.

But let us first clarify the question of temperature distribution.

Let us investigate an infinitely thin spherical layer with surface radii r and $r + dr$. In a stationary state the temperature will be $T(r)$. We will denote the coefficient of thermal conductivity by σ and assume it to be independent of r.

Through the surface $r + dr$ an amount of heat

$$-4\pi\sigma\,[r^2 T'(r)]_{r+dr} \tag{17}$$

will flow in unit time.

Through the surface r, an amount of heat

$$-4\pi\sigma\,[r^2 T'(r)]_r \tag{18}$$

will flow in unit time.

Consequently an amount of heat

$$4\pi\sigma\,\{[r^2T']_{r+dr} - [r^2T']_r\} \quad \text{or} \quad 4\pi\sigma\,[r^2T']'\,dr \tag{19}$$

will be added to the layer in unit time.

If this amount were not identically zero the state could not be stationary. The requirement of a stationary state leads to the Laplace equation (for spherical symmetry).

$$[r^2T']' = 0 \tag{20}$$

Its general solution is

$$T(r) = \frac{A}{r} + B \tag{21}$$

where A and B are arbitrary constants defined by boundary conditions.

For instance, let there be a layer between surfaces of radii R_1 and R_2 ($r_0 \leqslant R_1 < R_2$), maintained at a constant temperature T^* by a permanently acting radio-isotopic radiation uniformly distributed in this layer. Let a constant temperature T_0 be maintained on the surface r_0. Further, let $T(r) \to 0$ for $r \to \infty$. For this case a simple calculation will yield

$$T(r) = \frac{R_1 T^*(r - r_0) + r_0 T_0 (R_1 - r)}{(R_1 - r_0)\,r} \qquad (r_0 \leqslant r \leqslant R_1) \tag{22}$$

$$= T^* \qquad (R_1 \leqslant r \leqslant R_2) \tag{23}$$

$$= \frac{R_2 T^*}{r} \qquad (r \geqslant R) \tag{24}$$

In particular, if $R_2 = R_1 = r_0$ and $T^* = T_0$ and $T \to 0$ for $r \to \infty$, we will have a monotonic decrease of temperature from T_0 to 0, according to the hyperbolic law.

$$T(r) = \frac{r_0 T_0}{r} \qquad (r \geqslant r_0) \tag{25}$$

Subsequently we will confine ourselves to this particular temperature distribution.

From equation (16), on the basis of (13), we derive the following:

$$\rho(r) = \rho(r_0)\frac{r_0^2}{r^2}\exp\left[-\int_{r_0}^{r}\left[\frac{a}{r^2} + T'\right]\frac{dr}{T}\right] \tag{26}$$

According to (16) this yields

$$\rho(r) = \rho(r_0)\frac{r_0^2}{r^2}\exp\left[\ln\frac{r}{r_0} - \frac{a}{r_0 T_0}\ln\frac{r}{r_0}\right] \tag{27}$$

or

$$\rho(r) = \rho(r_0)\ \frac{r_0}{r}\exp\left[-\frac{a}{r_0 T_0}\ln\left(1 + \frac{r - r_0}{r_0}\right)\right] \tag{28}$$

Noting that $r - r_0 = h_0$ is the height above r_0, which is considered small by comparison with r_0, we can assume

$$\ln\left(1 + \frac{r - r_0}{r_0}\right) = \frac{h}{r_0} \tag{29}$$

Finally, making use of (16) and of the fact that

$$\frac{\gamma M_0 m}{r^2} = mg \tag{30}$$

represents the weight of the particle of mass m, instead of (28) we obtain

$$\rho(r) = \rho(r_0)\frac{r_0}{r}\exp\left[-\frac{mgh}{kT_0}\right] \tag{31}$$

Identification of r with r_0 leads to the Laplace formula (1).

Translated by J.S.